Geometric Modeling

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Subdivision

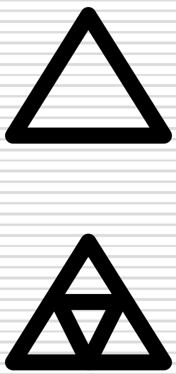
- What is Subdivision ?
- Subdivision in 1D & 2D
- Classification of Subdivision Schemes
- Doo-Sabin Surfaces
- Catmull-Clark Surfaces
- Loop's Scheme
- Butterfly Scheme

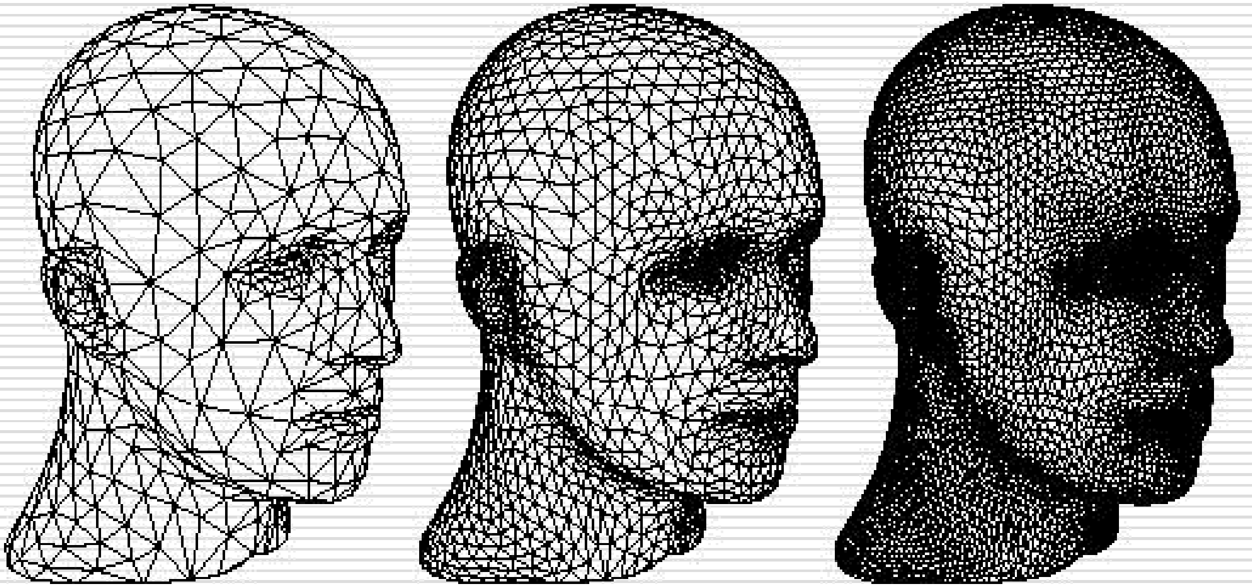
The Beginning

- create smooth surfaces out of arbitrary meshes
- the need to generalize spline patch modeling to arbitrary surfaces
  - Catmull-Clark
    - generalizes bi-cubic patches
  - Doo-Sabin
    - generalizes bi-quadratic patches

What is Subdivision ?

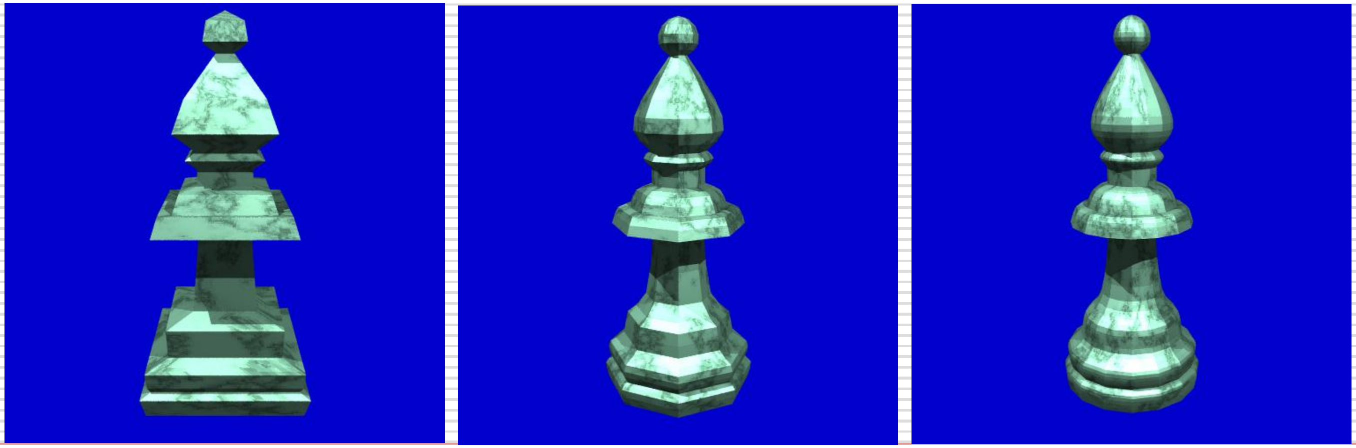
- defines smooth curve or surface as the limit of a sequence of successive refinements





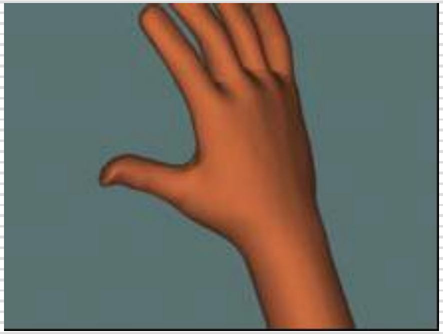
What is Subdivision ?

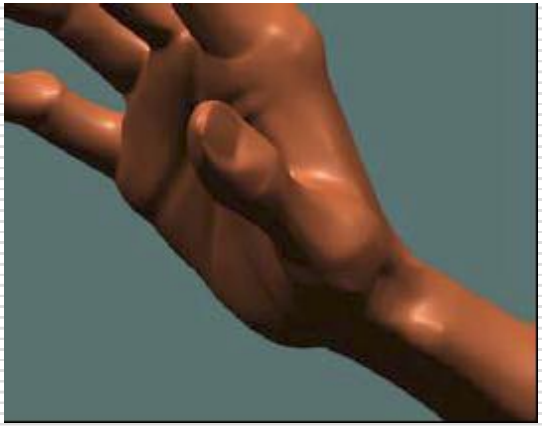
- given polyline(2D)/mesh(3D) recursively modify & add vertices to achieve smooth curve/surface
  - each iteration generates smoother + more refined mesh

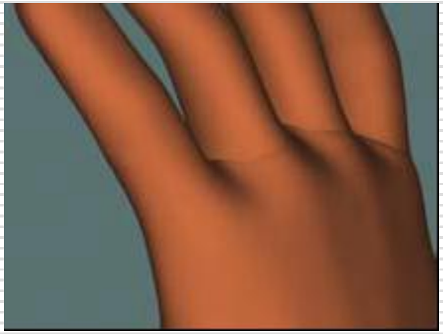


What is wrong with NURBS ?

- very, very difficult to avoid seams
  - Woody's hand from Toy Story
- Subdivision – no seams
  - Geri's hand from Geri's Game







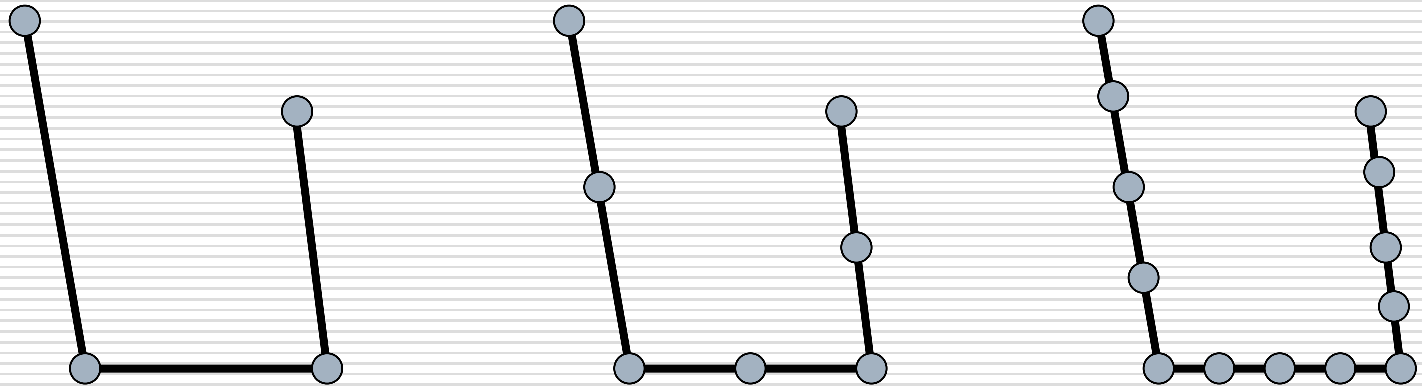
Why Subdivision ?

- many attractive features
  - arbitrary topology
  - scalability, LOD (Level-Of-Detail)
  - uniformity
  - numerical quality
  - code simplicity

Subdivision in 1D

- piecewise linear subdivision

$$x_n = \frac{1}{2}(x_l + x_r) \qquad y_n = \frac{1}{2}(y_l + y_r)$$

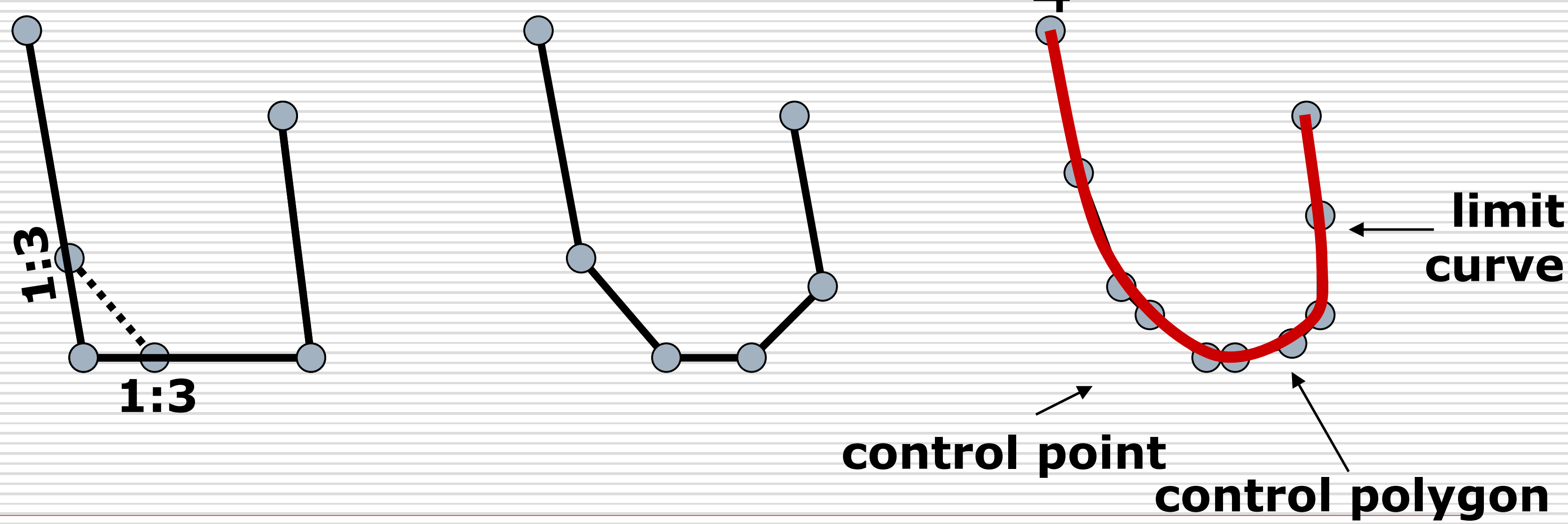


Subdivision in 1D

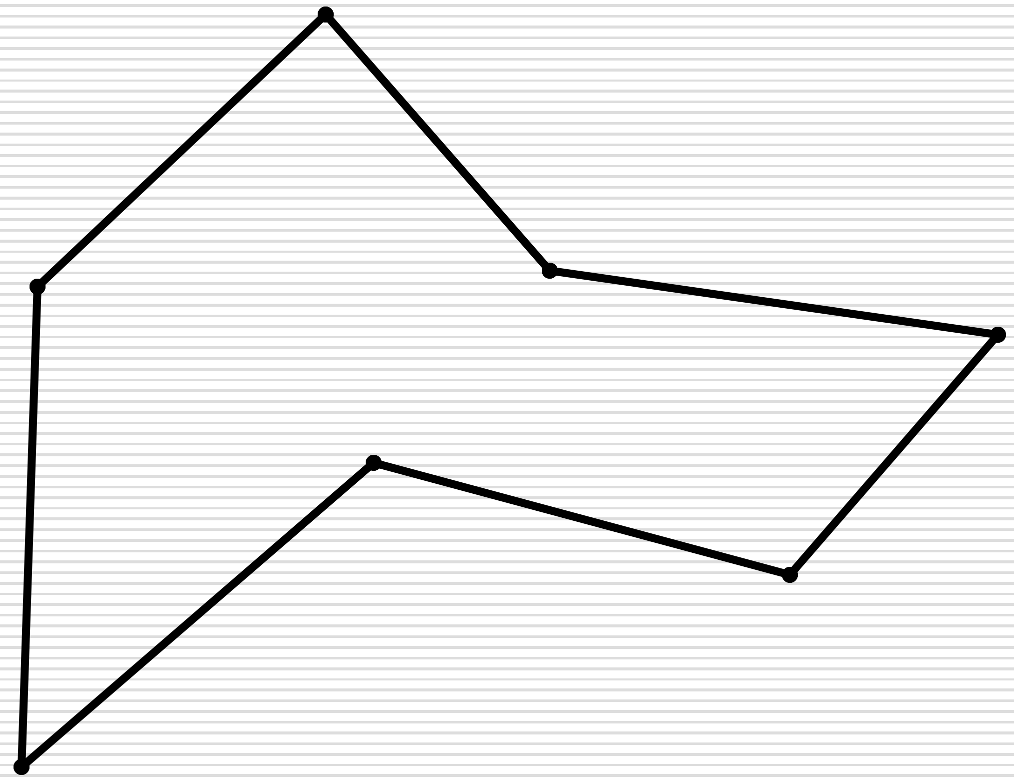
- corner cutting (Chaikin's method)

$$p_{2i-1}^{j+1} = \frac{1}{4}(p_{i-1}^{j} + 3p_{i}^{j})$$

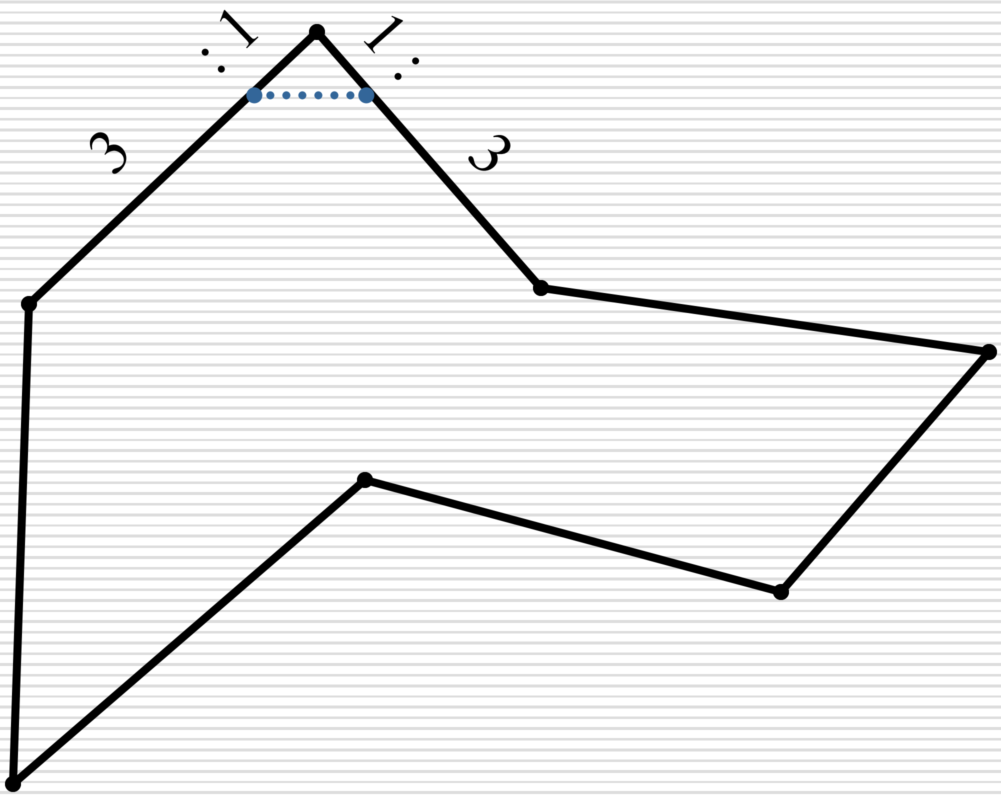
$$p_{2i}^{j+1} = \frac{1}{4}(3p_{i}^{j} + p_{i+1}^{j})$$



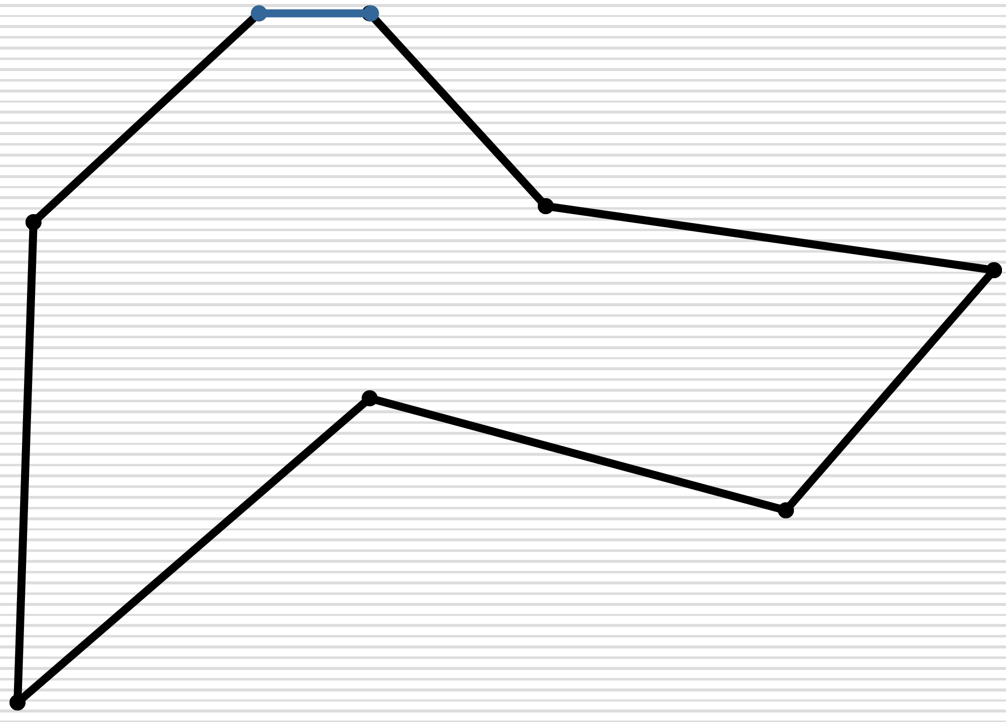
Approximation



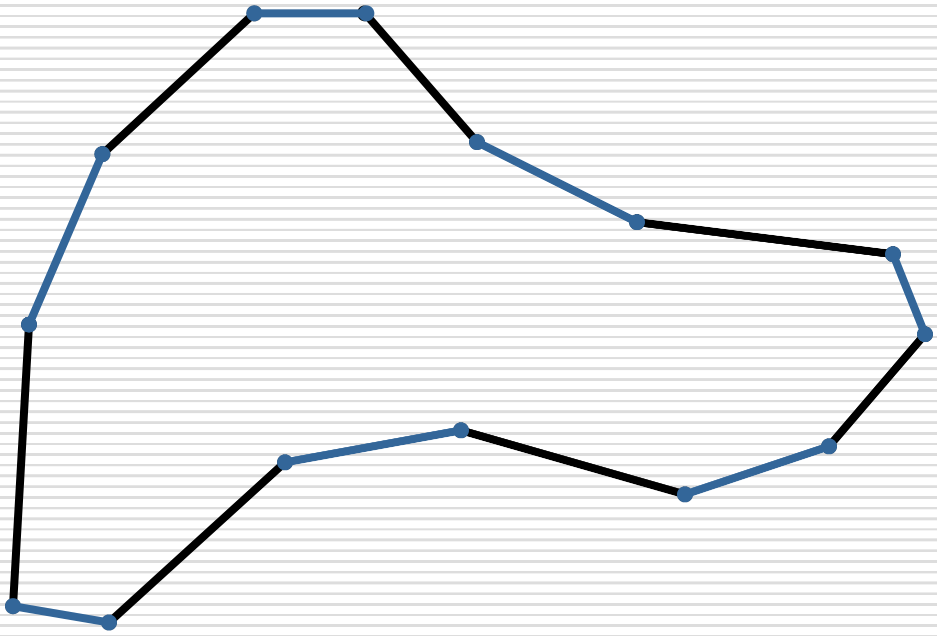
Approximation



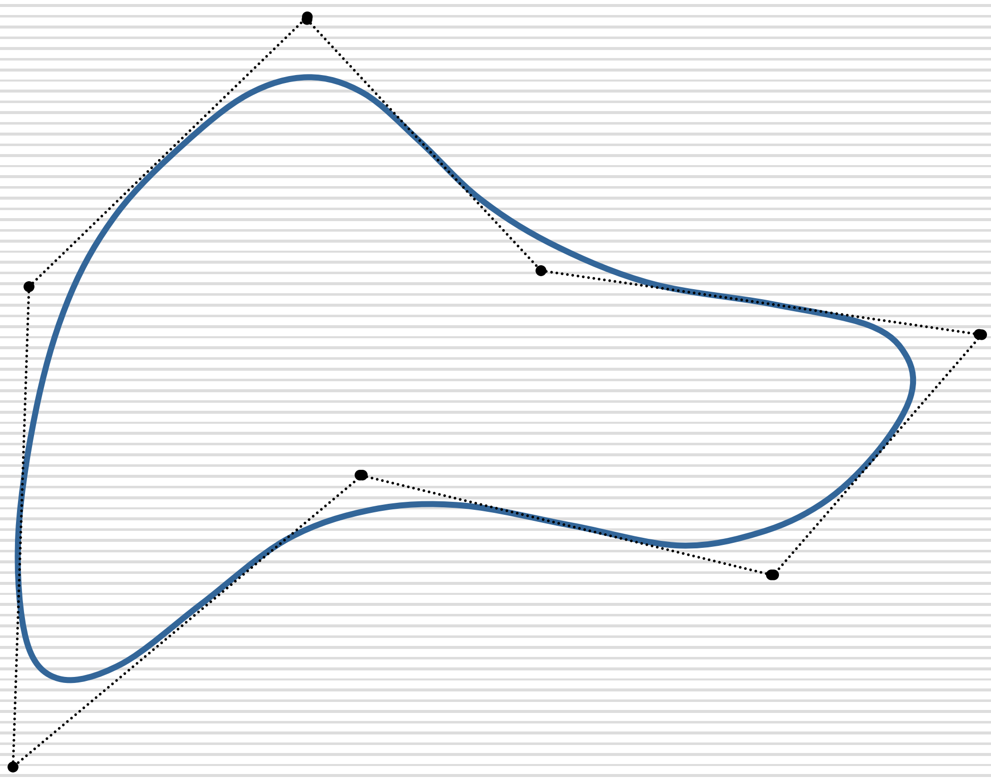
Approximation



Approximation



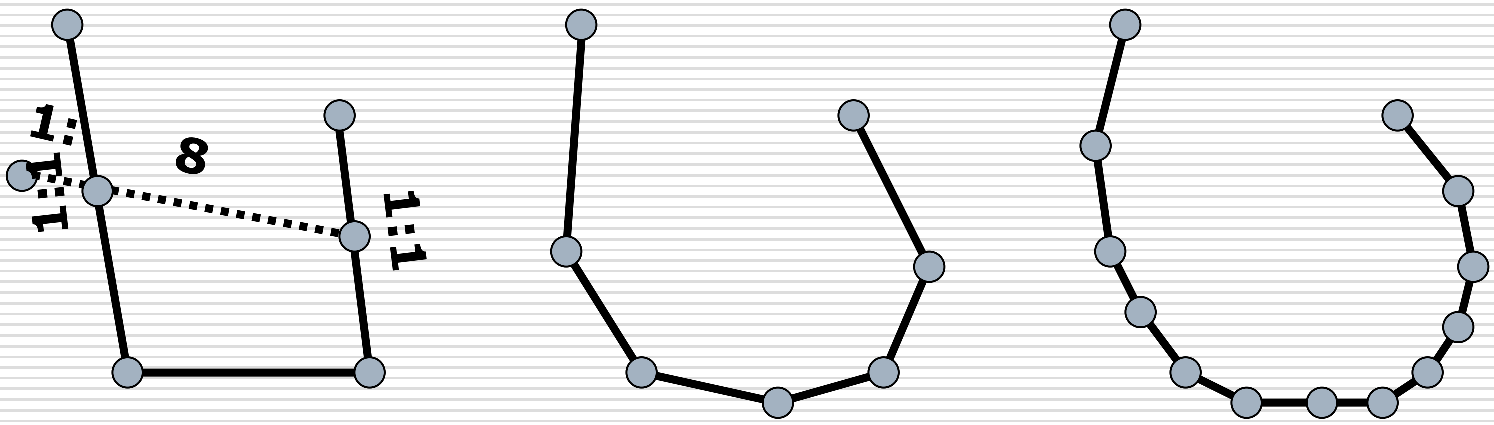
Approximation



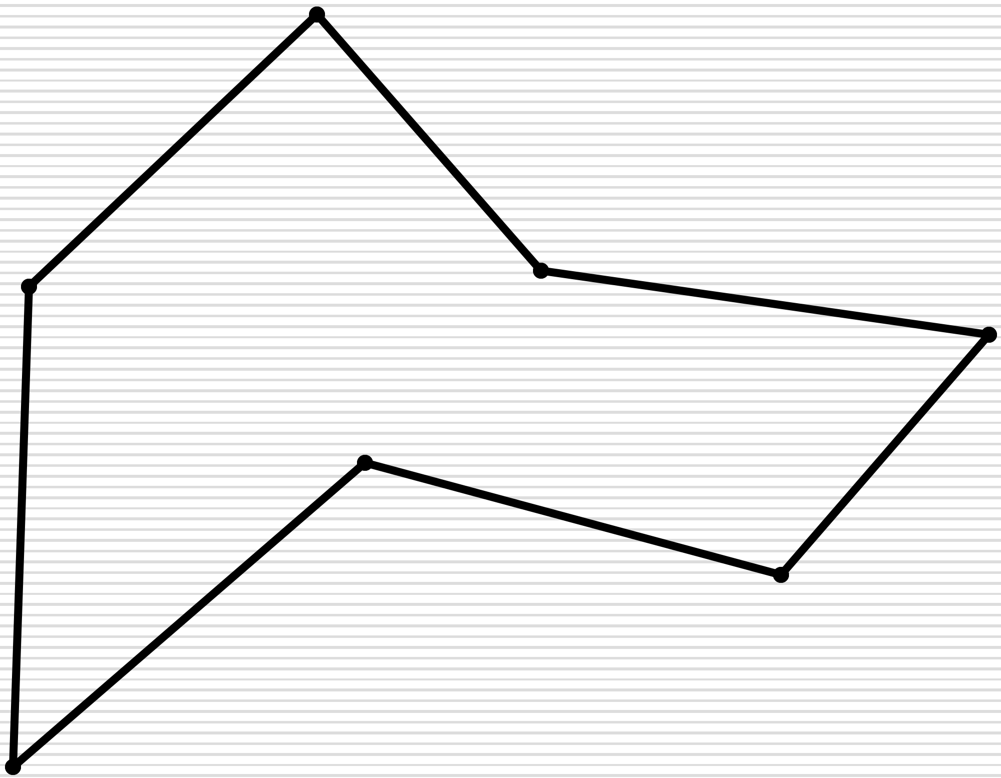
Subdivision in 1D

- the 4pt scheme

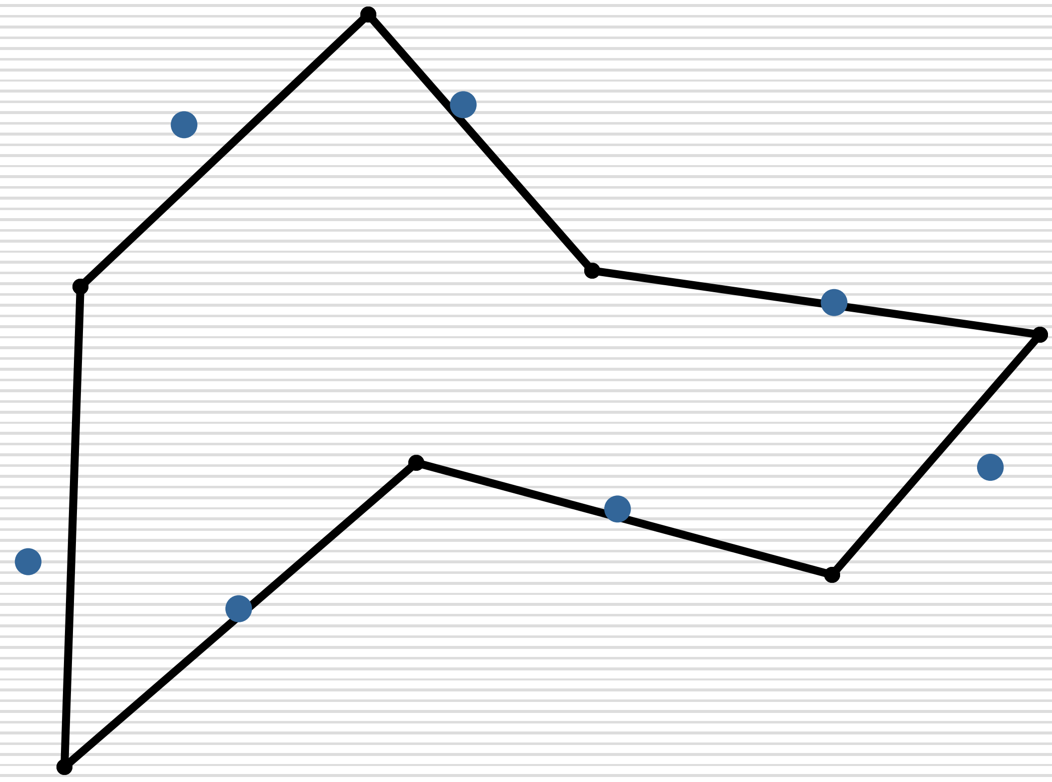
$$p_{2i+1}^{j+1} = \frac{1}{16}\left(-p_{i-1}^{j} + 9p_{i}^{j} + 9p_{i+1}^{j} - p_{i+2}^{j}\right)$$



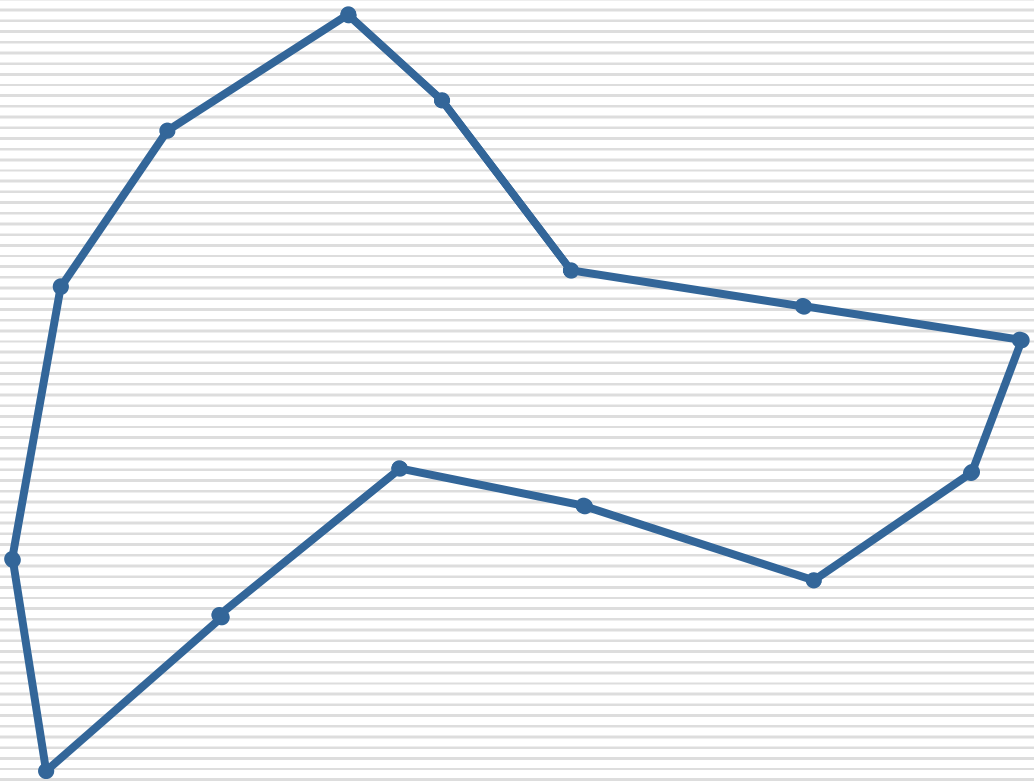
Interpolation



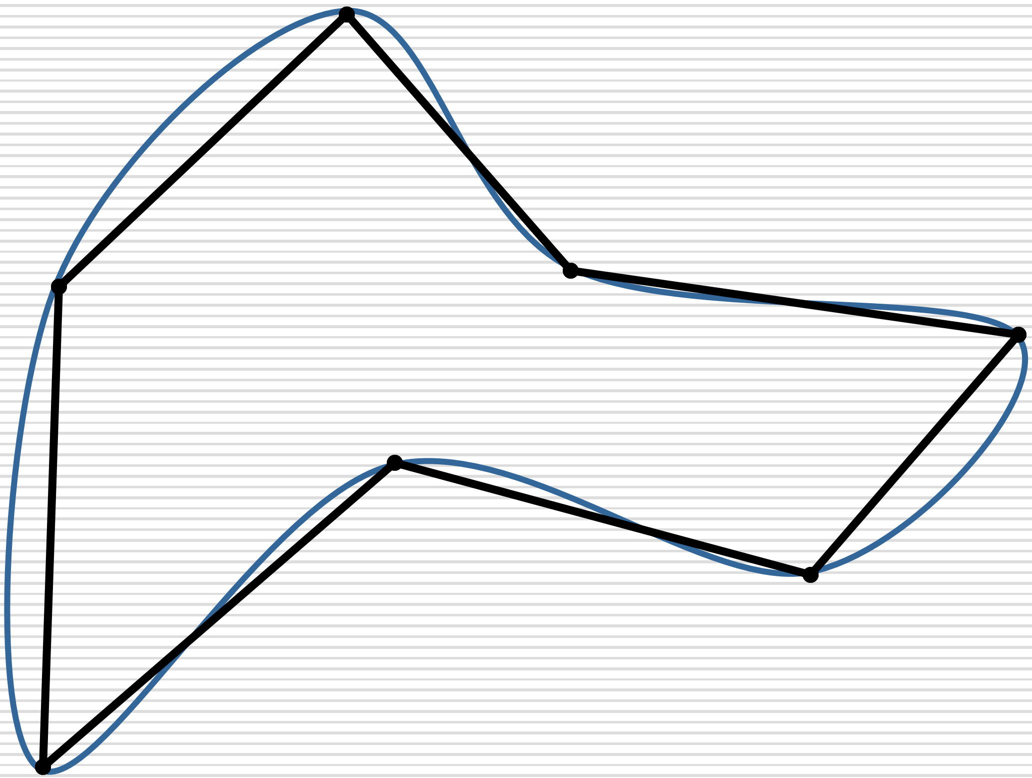
Interpolation



Interpolation



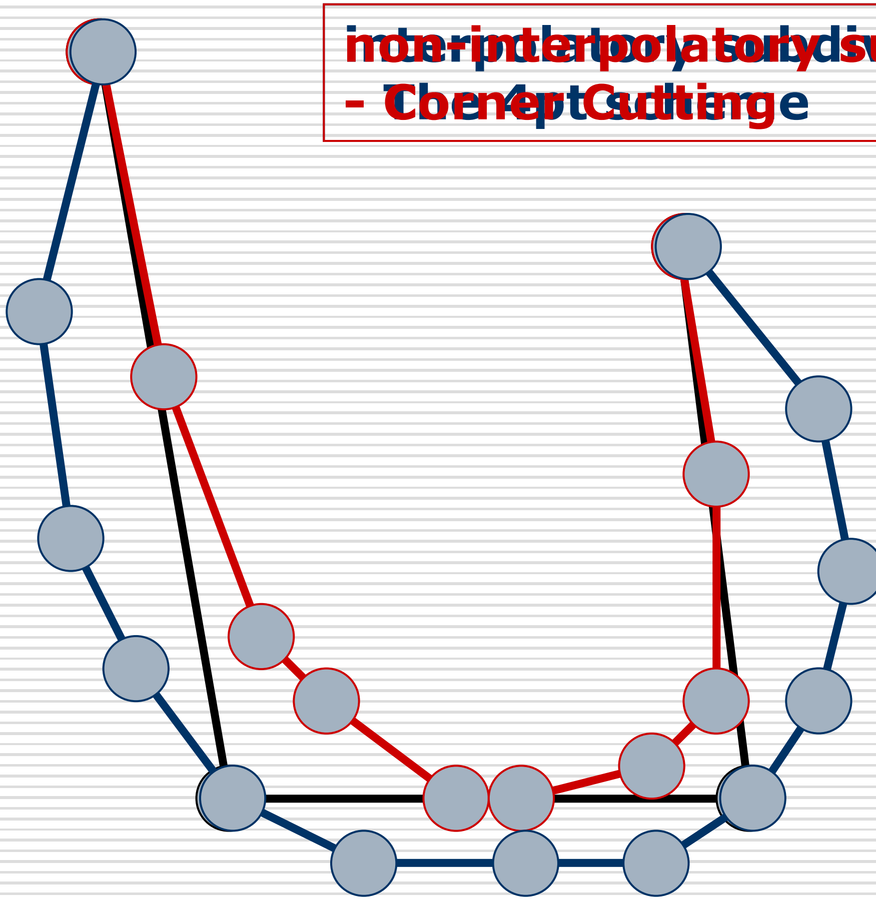
Interpolation



Subdivision Curves

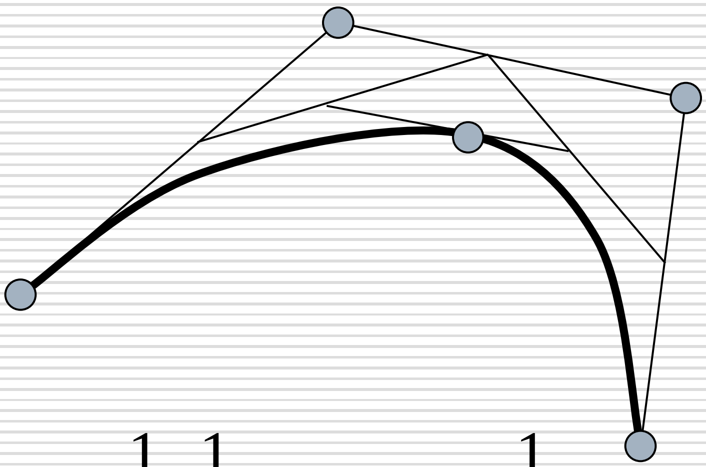
interpolatory subdivision schemes
- The 4pt scheme

non-interpolatory subdivision schemes
- Corner Cutting



Recall: Subdividing Bézier Curves

- $Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4$
- How to draw the curve ?
- How to convert it to be line-segments ?



$$Q(\frac{1}{2}) = \frac{1}{8}P_1 + \frac{3}{8}P_2 + \frac{3}{8}P_3 + \frac{1}{8}P_4$$

$$= \frac{1}{2}(\frac{1}{2}(\frac{1}{2}(P_1 + P_2) + \frac{1}{2}(P_2 + P_3)) + \frac{1}{2}(\frac{1}{2}(P_3 + P_4) + \frac{1}{2}(P_2 + P_3)))$$

Basic Concepts of Subdivision

- Subdivision curve
  - limit of recursive subdivision applied to given polygon
- Each iteration
  - increase number of vertices (approximately) * 2
- Initial polygon - <u>control polygon</u>
- Central questions:
  - <u>Convergence</u>:
    - Given a subdivision operator and a control polygon, does the subdivision process converge?
  - <u>Smoothness</u>:
    - Does subdivision converge to smooth curve?

Subdivision Schemes for Surfaces

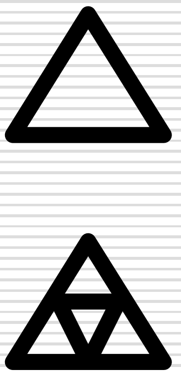
- Each iteration
  - subdivision refines control net (mesh)
  - increase number of vertices (approximately) * 4
- Mesh vertices converge to limit surface
- Every subdivision method has
  - method to generate net topology
  - rule to calculate location of new vertices

Classification of Schemes

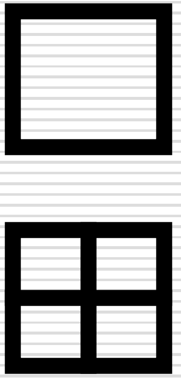
- classification criteria
  - type of refinement rule (primal or dual)
  - type of mesh (triangular or quad or…)
  - approximating or interpolating
  - smoothness of the limit surfaces for regular meshes (C^1, C^2…)

Subdivision in 2D

- Triangular
  - approximating: Loop scheme

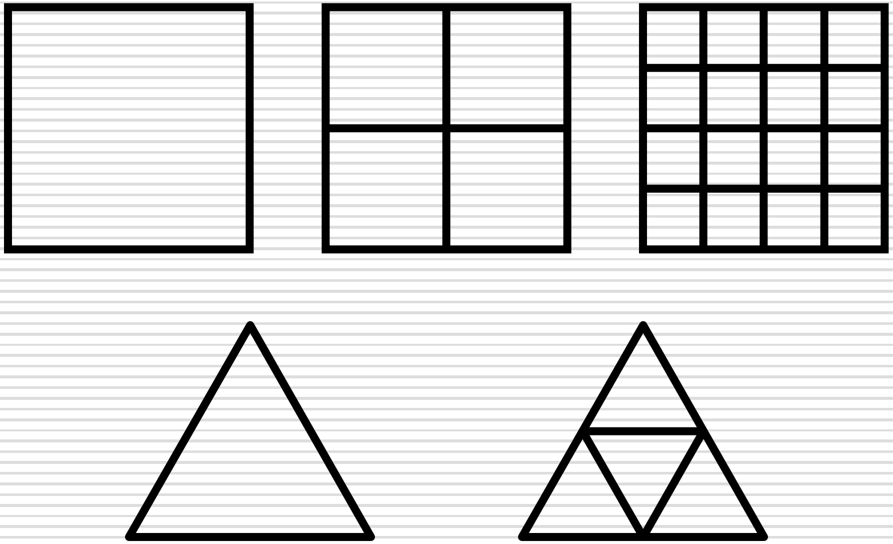


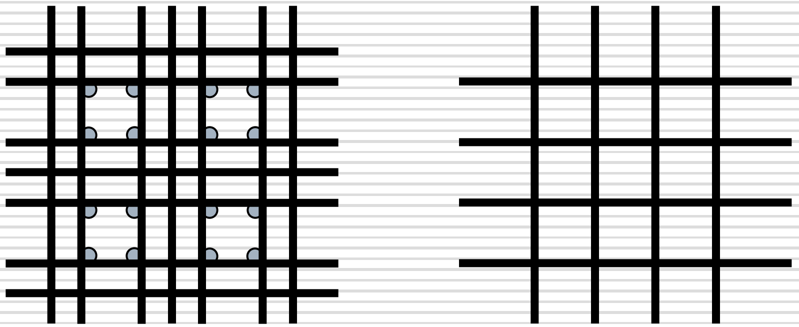
- Quadrilateral
  - interpolating: Kobbelt scheme



Refinement Rule

- primal
  - vertex insertion
  - face split
- dual
  - corner cutting
  - vertex split





Approximating & Interpolating

- Approximating
  - Vertices are moved
  - Converges into limit surface
  - Convergence faster than interpolating schemes
  - Limit Surface will not necessarily pass through the original set of data points

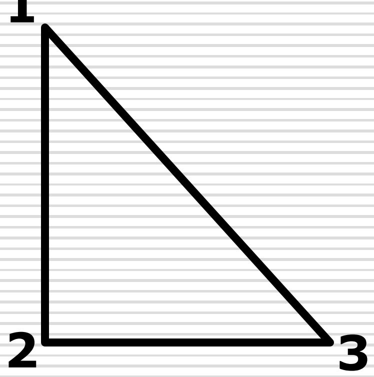
Approximating & Interpolating

- Interpolating
  - Only insert new vertices, old vertices remain stationary
  - ‘Inflate’ to limit surface
  - More sensitive to sharp features
  - Limit Surfaces will pass through original set of data points.
    - Butterfly (tri based)
    - Kabbelt (quad based)

Data Structure of Subdivision Surfaces

- A mesh is a pair (K,V), where K is a **simplicial complex** representing connectivity of the vertices, edges, and faces (topological type) and V is a set of vertex positions defining the shape of the mesh.

- Example:
  - V={[0,1,0],[0,0,0],[1,0,0]}
  - simplicial complex K:
    - vertices: {1},{2},{3}
    - edges: {1,2},{2,3},{1,3}
    - faces: {1,2,3}



Recall: Uniform B-Spline Surface

- Parametric surface is represented as

$$Q(s,t) = T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet S, \qquad 0 \le s,t \le 1$$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{21} & \boldsymbol{g}_{31} \\ \boldsymbol{g}_{12} & \boldsymbol{g}_{22} & \boldsymbol{g}_{32} \\ \boldsymbol{g}_{13} & \boldsymbol{g}_{23} & \boldsymbol{g}_{33} \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{21} & \boldsymbol{g}_{31} & \boldsymbol{g}_{41} \\ \boldsymbol{g}_{12} & \boldsymbol{g}_{22} & \boldsymbol{g}_{32} & \boldsymbol{g}_{42} \\ \boldsymbol{g}_{13} & \boldsymbol{g}_{23} & \boldsymbol{g}_{33} & \boldsymbol{g}_{43} \\ \boldsymbol{g}_{14} & \boldsymbol{g}_{24} & \boldsymbol{g}_{34} & \boldsymbol{g}_{44} \end{bmatrix}$$

$$M = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bi-Quadratic **Bi-Cubic**

Subdividing Uniform B-Spline Surface

- To subdivide the surface, we consider the re-parameterization of the surface by $s' = \frac{s}{2}, t' = \frac{t}{2}$ and define this new surface as

$$Q'(s,t) = Q(\frac{s}{2}, \frac{t}{2})$$

$$= T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G'} \bullet M \bullet S$$

where $\boldsymbol{G'} = M'^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M'$

$$M' = \left(M^{-1} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \bullet M \right)$$

Bi-Quadratic

or

$$M' = \left(M^{-1} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \bullet M \right)$$

Bi-Cubic

Subdividing Bi-Quadratic Uniform B-Spline Surface

$$\boldsymbol{G}' = \frac{1}{4}\begin{bmatrix} 0 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{21} & \boldsymbol{g}_{31} \\ \boldsymbol{g}_{12} & \boldsymbol{g}_{22} & \boldsymbol{g}_{32} \\ \boldsymbol{g}_{13} & \boldsymbol{g}_{23} & \boldsymbol{g}_{33} \end{bmatrix} \bullet \frac{1}{4}\begin{bmatrix} 0 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} \boldsymbol{g}'_{11} & \boldsymbol{g}'_{21} & \boldsymbol{g}'_{31} \\ \boldsymbol{g}'_{12} & \boldsymbol{g}'_{22} & \boldsymbol{g}'_{32} \\ \boldsymbol{g}'_{13} & \boldsymbol{g}'_{23} & \boldsymbol{g}'_{33} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{g}'_{11} = \frac{1}{16}(3(3\boldsymbol{g}_{11} + \boldsymbol{g}_{21}) + (3\boldsymbol{g}_{12} + \boldsymbol{g}_{22}))$$

$$\boldsymbol{g}'_{12} = \frac{1}{16}((3\boldsymbol{g}_{11} + \boldsymbol{g}_{21}) + 3(3\boldsymbol{g}_{12} + \boldsymbol{g}_{22}))$$

$$\boldsymbol{g}'_{13} = \frac{1}{16}(3(3\boldsymbol{g}_{12} + \boldsymbol{g}_{22}) + (3\boldsymbol{g}_{13} + \boldsymbol{g}_{23}))$$

$$\boldsymbol{g}'_{21} = \frac{1}{16}(3(\boldsymbol{g}_{11} + 3\boldsymbol{g}_{21}) + (\boldsymbol{g}_{12} + 3\boldsymbol{g}_{22}))$$

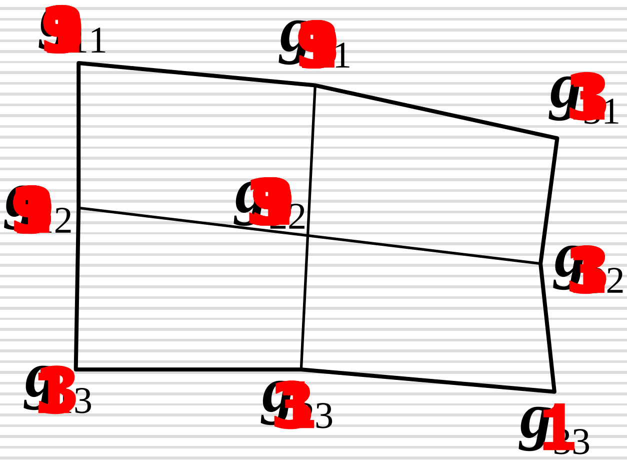
$$\boldsymbol{g}'_{22} = \frac{1}{16}((\boldsymbol{g}_{11} + 3\boldsymbol{g}_{21}) + 3(\boldsymbol{g}_{12} + 3\boldsymbol{g}_{22}))$$

$$\boldsymbol{g}'_{23} = \frac{1}{16}(3(\boldsymbol{g}_{12} + 3\boldsymbol{g}_{22}) + (\boldsymbol{g}_{13} + 3\boldsymbol{g}_{23}))$$

$$\boldsymbol{g}'_{31} = \frac{1}{16}(3(3\boldsymbol{g}_{21} + \boldsymbol{g}_{31}) + (3\boldsymbol{g}_{22} + \boldsymbol{g}_{32}))$$

$$\boldsymbol{g}'_{32} = \frac{1}{16}((3\boldsymbol{g}_{21} + \boldsymbol{g}_{31}) + 3(3\boldsymbol{g}_{22} + \boldsymbol{g}_{32}))$$

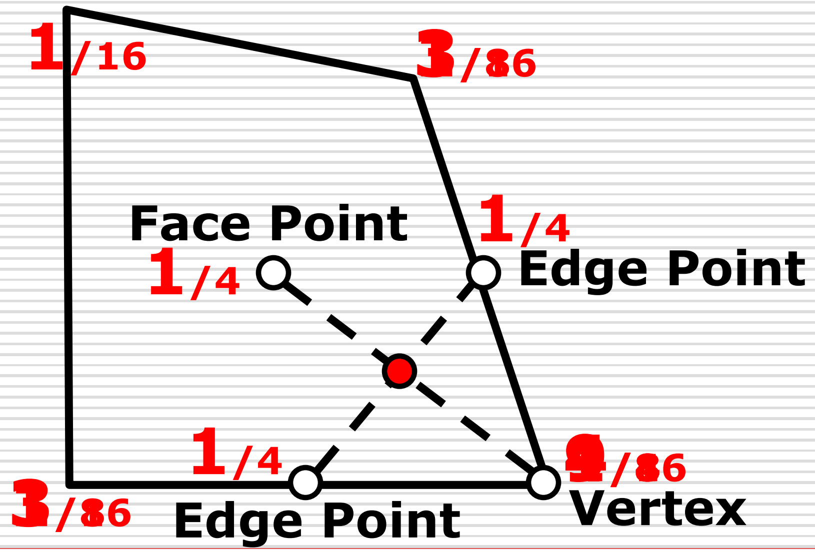
$$\boldsymbol{g}'_{33} = \frac{1}{16}(3(3\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + (3\boldsymbol{g}_{23} + \boldsymbol{g}_{33}))$$



Doo-Sabin Surfaces

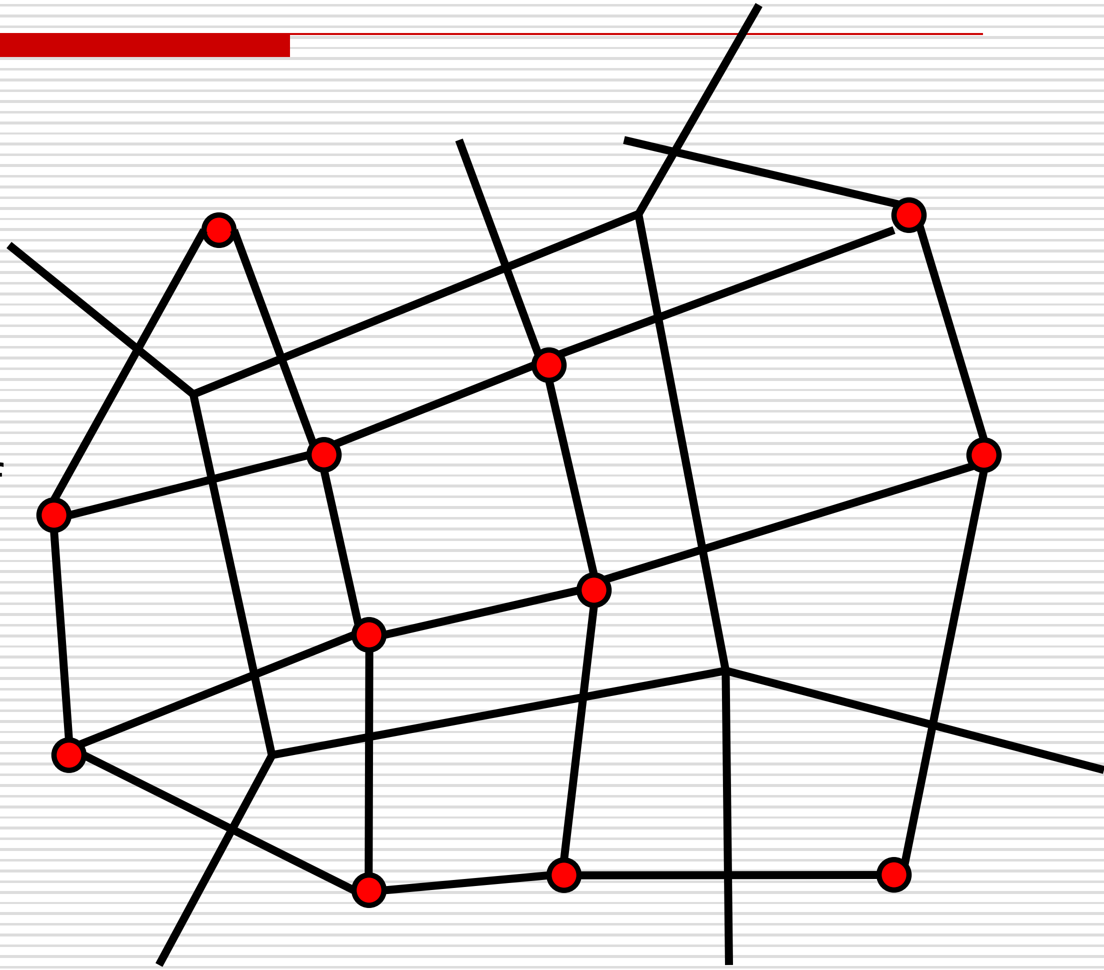
- subdivision masks

9 — 3	3 — 1	1 — 3	3 — 9
| |	| |	| |	| |
3 — 1	9 — 3	3 — 9	1 — 3

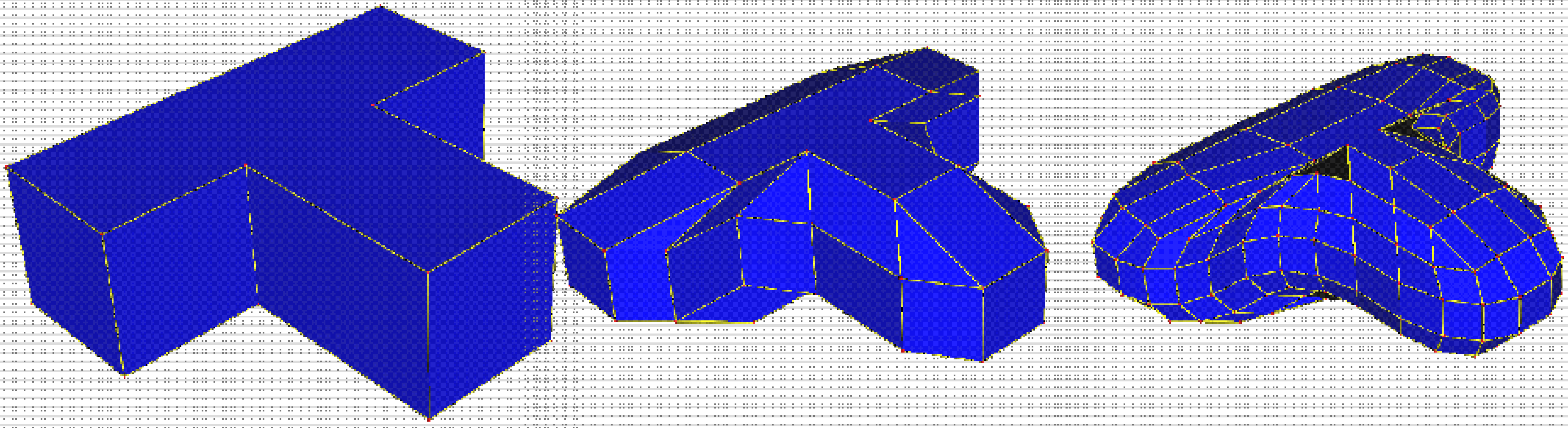


Doo-Sabin Surfaces

1. For each vertex of each face of the object, generate a new point as average of the vertex, the two edge points and the face point of the face.
2. For each face, connect the new points that have been generated for each vertex of the face.
3. For each vertex, connect the new points that have been generated for the faces that are adjacent to this vertex.
4. For each edge, connect the new points that have been generated for the faces that are adjacent to this edge.



Doo-Sabin Surfaces



Subdividing
Bi-Cubic Uniform B-Spline Surface

$$\boldsymbol{G}' = \frac{1}{8}\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 4 & 6 & 4 \\ 6 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{T} \bullet \begin{bmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{21} & \boldsymbol{g}_{31} & \boldsymbol{g}_{41} \\ \boldsymbol{g}_{12} & \boldsymbol{g}_{22} & \boldsymbol{g}_{32} & \boldsymbol{g}_{42} \\ \boldsymbol{g}_{13} & \boldsymbol{g}_{23} & \boldsymbol{g}_{33} & \boldsymbol{g}_{43} \\ \boldsymbol{g}_{14} & \boldsymbol{g}_{24} & \boldsymbol{g}_{34} & \boldsymbol{g}_{44} \end{bmatrix} \bullet \frac{1}{8}\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 4 & 6 & 4 \\ 6 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{64}\begin{bmatrix} \boldsymbol{g}'_{11} & \boldsymbol{g}'_{21} & \boldsymbol{g}'_{31} & \boldsymbol{g}'_{41} \\ \boldsymbol{g}'_{12} & \boldsymbol{g}'_{22} & \boldsymbol{g}'_{32} & \boldsymbol{g}'_{42} \\ \boldsymbol{g}'_{13} & \boldsymbol{g}'_{23} & \boldsymbol{g}'_{33} & \boldsymbol{g}'_{43} \\ \boldsymbol{g}'_{14} & \boldsymbol{g}'_{24} & \boldsymbol{g}'_{34} & \boldsymbol{g}'_{44} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{g}'_{11} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + \boldsymbol{g}_{22}}{4}$$

$$\boldsymbol{g}'_{12} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + 6(\boldsymbol{g}_{12} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{16}$$

$$\boldsymbol{g}'_{13} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{4}$$

$$\boldsymbol{g}'_{14} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{13} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{14} + \boldsymbol{g}_{24}}{16}$$

$$\boldsymbol{g}'_{21} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{12} + 6(\boldsymbol{g}_{21} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{31} + \boldsymbol{g}_{32}}{16}$$

$$\boldsymbol{g}'_{22} = \frac{\boldsymbol{g}_{11} + 6\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{64}$$

$$\boldsymbol{g}'_{23} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{13} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{32} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{24} = \frac{\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{14} + 6\boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{64}$$

$$\boldsymbol{g}'_{31} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + \boldsymbol{g}_{22} + \boldsymbol{g}_{32}}{4}$$

$$\boldsymbol{g}'_{32} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{33} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{4}$$

$$\boldsymbol{g}'_{34} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{16}$$

$$\boldsymbol{g}'_{41} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{31} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{41} + \boldsymbol{g}_{42}}{16}$$

$$\boldsymbol{g}'_{42} = \frac{\boldsymbol{g}_{21} + 6\boldsymbol{g}_{31} + \boldsymbol{g}_{41} + 6(\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42}) + \boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}}{64}$$

$$\boldsymbol{g}'_{43} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{23} + 6(\boldsymbol{g}_{32} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{42} + \boldsymbol{g}_{43}}{16}$$

$$\boldsymbol{g}'_{44} = \frac{\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42} + 6(\boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}) + \boldsymbol{g}_{24} + 6\boldsymbol{g}_{34} + \boldsymbol{g}_{44}}{64}$$

Subdividing Bi-Cubic Uniform B-Spline Surface

$$\boldsymbol{g}'_{11} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + \boldsymbol{g}_{22}}{4}$$

$$\boldsymbol{g}'_{12} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + 6(\boldsymbol{g}_{12} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{16}$$

$$\boldsymbol{g}'_{13} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{4}$$

$$\boldsymbol{g}'_{14} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{13} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{14} + \boldsymbol{g}_{24}}{16}$$

$$\boldsymbol{g}'_{21} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{12} + 6(\boldsymbol{g}_{21} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{31} + \boldsymbol{g}_{32}}{16}$$

$$\boldsymbol{g}'_{22} = \frac{\boldsymbol{g}_{11} + 6\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{64}$$

$$\boldsymbol{g}'_{23} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{13} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{32} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{24} = \frac{\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{14} + 6\boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{64}$$

$$\boldsymbol{g}'_{31} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + \boldsymbol{g}_{22} + \boldsymbol{g}_{32}}{4}$$

$$\boldsymbol{g}'_{32} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{33} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{4}$$

$$\boldsymbol{g}'_{34} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{16}$$

$$\boldsymbol{g}'_{41} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{31} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{41} + \boldsymbol{g}_{42}}{16}$$

$$\boldsymbol{g}'_{42} = \frac{\boldsymbol{g}_{21} + 6\boldsymbol{g}_{31} + \boldsymbol{g}_{41} + 6(\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42}) + \boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}}{64}$$

$$\boldsymbol{g}'_{43} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{23} + 6(\boldsymbol{g}_{32} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{42} + \boldsymbol{g}_{43}}{16}$$

$$\boldsymbol{g}'_{44} = \frac{\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42} + 6(\boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}) + \boldsymbol{g}_{24} + 6\boldsymbol{g}_{34} + \boldsymbol{g}_{44}}{64}$$

Subdividing Bi-Cubic Uniform B-Spline Surface

$$\boldsymbol{g}'_{11} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + \boldsymbol{g}_{22}}{4}$$

$$\boldsymbol{g}'_{12} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + 6(\boldsymbol{g}_{12} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{16}$$

$$\boldsymbol{g}'_{13} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{4}$$

$$\boldsymbol{g}'_{14} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{13} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{14} + \boldsymbol{g}_{24}}{16}$$

$$\boldsymbol{g}'_{21} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{12} + 6(\boldsymbol{g}_{21} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{31} + \boldsymbol{g}_{32}}{16}$$

$$\boldsymbol{g}'_{22} = \frac{\boldsymbol{g}_{11} + 6\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{64}$$

$$\boldsymbol{g}'_{23} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{13} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{32} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{24} = \frac{\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{14} + 6\boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{64}$$

$$\boldsymbol{g}'_{31} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + \boldsymbol{g}_{22} + \boldsymbol{g}_{32}}{4}$$

$$\boldsymbol{g}'_{32} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{33} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + \boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{4}$$

$$\boldsymbol{g}'_{34} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{16}$$

$$\boldsymbol{g}'_{41} = \frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{31} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{41} + \boldsymbol{g}_{42}}{16}$$

$$\boldsymbol{g}'_{42} = \frac{\boldsymbol{g}_{21} + 6\boldsymbol{g}_{31} + \boldsymbol{g}_{41} + 6(\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42}) + \boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}}{64}$$

$$\boldsymbol{g}'_{43} = \frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{23} + 6(\boldsymbol{g}_{32} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{42} + \boldsymbol{g}_{43}}{16}$$

$$\boldsymbol{g}'_{44} = \frac{\boldsymbol{g}_{22} + 6\boldsymbol{g}_{32} + \boldsymbol{g}_{42} + 6(\boldsymbol{g}_{23} + 6\boldsymbol{g}_{33} + \boldsymbol{g}_{43}) + \boldsymbol{g}_{24} + 6\boldsymbol{g}_{34} + \boldsymbol{g}_{44}}{64}$$

$$\Rightarrow$$

$$\boldsymbol{g}'_{11} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + \boldsymbol{g}_{22}}{4}$$

$$\boldsymbol{g}'_{12} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{21} + 6(\boldsymbol{g}_{12} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{16}$$

$$\boldsymbol{g}'_{13} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{4}$$

$$\boldsymbol{g}'_{14} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22} + 6(\boldsymbol{g}_{13} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{14} + \boldsymbol{g}_{24}}{16}$$

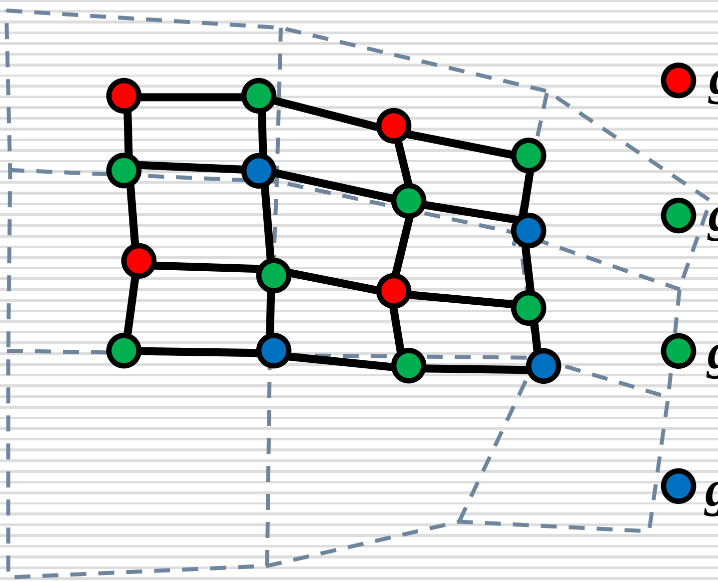
$$\boldsymbol{g}'_{21} = \frac{\boldsymbol{g}_{11} + \boldsymbol{g}_{12} + 6(\boldsymbol{g}_{21} + \boldsymbol{g}_{22}) + \boldsymbol{g}_{31} + \boldsymbol{g}_{32}}{16}$$

$$\boldsymbol{g}'_{22} = \frac{\boldsymbol{g}_{11} + 6\boldsymbol{g}_{21} + \boldsymbol{g}_{31} + 6(\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32}) + \boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}}{64}$$

$$\boldsymbol{g}'_{23} = \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{13} + 6(\boldsymbol{g}_{22} + \boldsymbol{g}_{23}) + \boldsymbol{g}_{32} + \boldsymbol{g}_{33}}{16}$$

$$\boldsymbol{g}'_{24} = \frac{\boldsymbol{g}_{12} + 6\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + 6(\boldsymbol{g}_{13} + 6\boldsymbol{g}_{23} + \boldsymbol{g}_{33}) + \boldsymbol{g}_{14} + 6\boldsymbol{g}_{24} + \boldsymbol{g}_{34}}{64}$$

Subdividing
Bi-Cubic Uniform B-Spline Surface

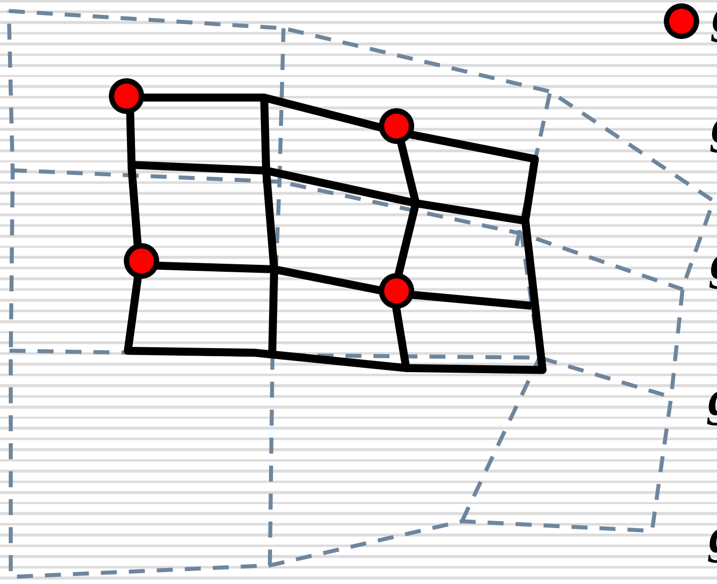


- $g'_{11} = \dfrac{g_{11} + g_{21} + g_{12} + g_{22}}{4}$
- $g'_{12} = \dfrac{g_{11} + g_{21} + 6(g_{12} + g_{22}) + g_{13} + g_{23}}{16}$
- $g'_{13} = \dfrac{g_{12} + g_{22} + g_{13} + g_{23}}{4}$
- $g'_{14} = \dfrac{g_{12} + g_{22} + 6(g_{13} + g_{23}) + g_{14} + g_{24}}{16}$
- $g'_{21} = \dfrac{g_{11} + g_{12} + 6(g_{21} + g_{22}) + g_{31} + g_{32}}{16}$
- $g'_{22} = \dfrac{g_{11} + 6g_{21} + g_{31} + 6(g_{12} + 6g_{22} + g_{32}) + g_{13} + 6g_{23} + g_{33}}{64}$
- $g'_{23} = \dfrac{g_{12} + g_{13} + 6(g_{22} + g_{23}) + g_{32} + g_{33}}{16}$
- $g'_{24} = \dfrac{g_{12} + 6g_{22} + g_{32} + 6(g_{13} + 6g_{23} + g_{33}) + g_{14} + 6g_{24} + g_{34}}{64}$

Subdividing
Bi-Cubic Uniform B-Spline Surface

$$\frac{\boldsymbol{g}_{ij} + \boldsymbol{g}_{(i+1)j} + \boldsymbol{g}_{i(j+1)} + \boldsymbol{g}_{(i+1)(j+1)}}{4} = \mathbf{F}_{ij}$$

Face Point



$$\boldsymbol{g}'_{11} = \mathbf{F}_{11}$$

$$\boldsymbol{g}'_{12} = \frac{\mathbf{F}_{11} + \mathbf{F}_{12} + (\boldsymbol{g}_{12} + \boldsymbol{g}_{22})}{4}$$

$$\boldsymbol{g}'_{13} = \mathbf{F}_{12}$$

$$\boldsymbol{g}'_{14} = \frac{\mathbf{F}_{12} + \mathbf{F}_{13} + (\boldsymbol{g}_{13} + \boldsymbol{g}_{23})}{4}$$

$$\boldsymbol{g}'_{21} = \frac{\mathbf{F}_{11} + \mathbf{F}_{21} + (\boldsymbol{g}_{21} + \boldsymbol{g}_{22})}{4}$$

$$\boldsymbol{g}'_{22} = \frac{\mathbf{F}_{11} + \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}_{22} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + 8\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + \boldsymbol{g}_{23}}{16}$$

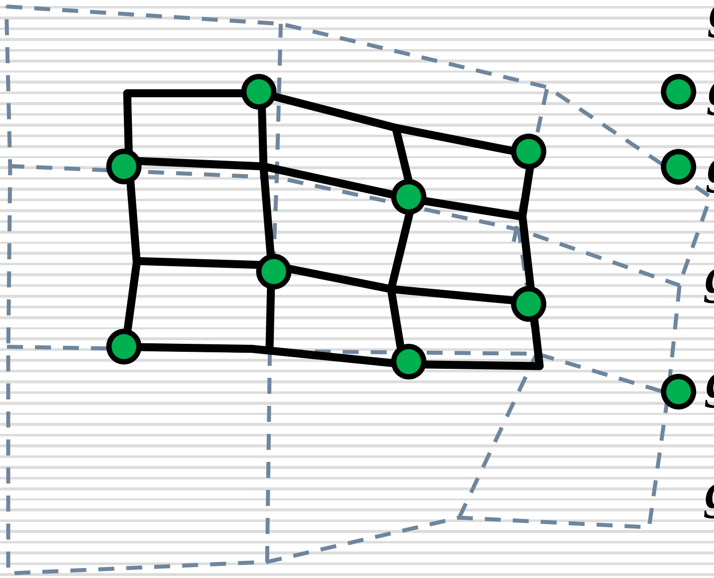
$$\boldsymbol{g}'_{23} = \frac{\mathbf{F}_{12} + \mathbf{F}_{22} + (\boldsymbol{g}_{22} + \boldsymbol{g}_{23})}{4}$$

$$\boldsymbol{g}'_{24} = \frac{\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{22} + \mathbf{F}_{23} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + 8\boldsymbol{g}_{23} + \boldsymbol{g}_{33} + 4\boldsymbol{g}_{24}}{16}$$

Subdividing Bi-Cubic Uniform B-Spline Surface

$$\frac{\mathbf{F}_{i(j-1)} + \mathbf{F}_{ij} + (\boldsymbol{g}_{ij} + \boldsymbol{g}_{(i+1)j})}{4} = \mathbf{E}_{ij}$$

Edge Point



$$\boldsymbol{g}'_{11} = \mathbf{F}_{11}$$

$$\boldsymbol{g}'_{12} = \mathbf{E}_{12}$$

$$\boldsymbol{g}'_{13} = \mathbf{F}_{12}$$

$$\boldsymbol{g}'_{14} = \mathbf{E}_{13}$$

$$\boldsymbol{g}'_{21} = \mathbf{E}_{21}$$

$$\boldsymbol{g}'_{22} = \frac{\mathbf{F}_{11} + \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}_{22} + \boldsymbol{g}_{21} + \boldsymbol{g}_{12} + 8\boldsymbol{g}_{22} + \boldsymbol{g}_{32} + \boldsymbol{g}_{23}}{16}$$

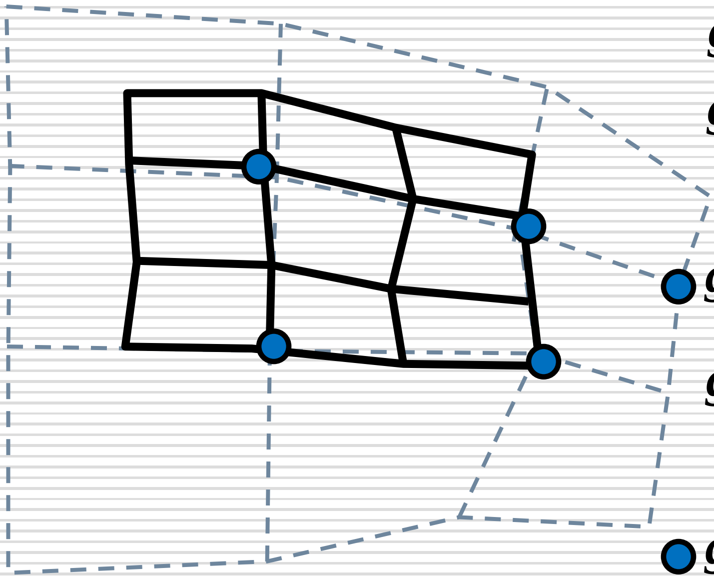
$$\boldsymbol{g}'_{23} = \mathbf{E}_{23}$$

$$\boldsymbol{g}'_{24} = \frac{\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{22} + \mathbf{F}_{23} + \boldsymbol{g}_{22} + \boldsymbol{g}_{13} + 8\boldsymbol{g}_{23} + \boldsymbol{g}_{33} + 4\boldsymbol{g}_{24}}{16}$$

$$\frac{\mathbf{F}_{(i-1)j} + \mathbf{F}_{ij} + (\boldsymbol{g}_{ij} + \boldsymbol{g}_{i(j+1)})}{4} = \mathbf{E}_{ij}$$

Subdividing
Bi-Cubic Uniform B-Spline Surface

Vertex Point



$$\boldsymbol{g}'_{11} = \mathbf{F}_{11}$$

$$\boldsymbol{g}'_{12} = \mathbf{E}_{12}$$

$$\boldsymbol{g}'_{13} = \mathbf{F}_{12}$$

$$\boldsymbol{g}'_{14} = \mathbf{E}_{13}$$

$$\boldsymbol{g}'_{21} = \mathbf{E}_{21}$$

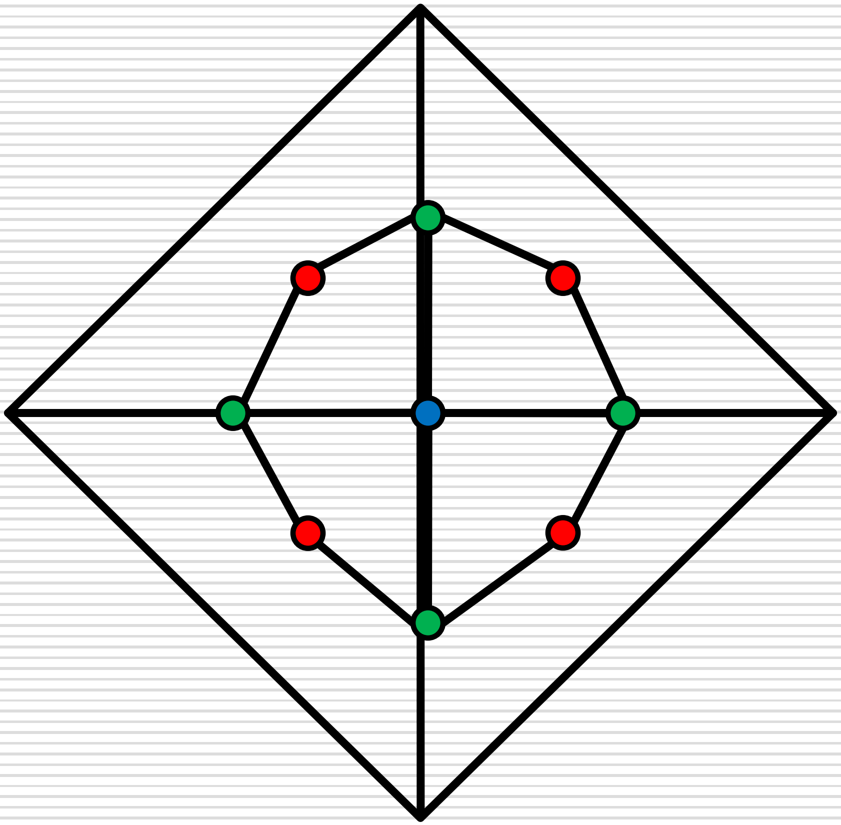
$$\boldsymbol{g}'_{22} = \frac{\mathbf{F}_{11} + \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}_{22}}{4} + \frac{\frac{\boldsymbol{g}_{21} + \boldsymbol{g}_{22}}{2} + \frac{\boldsymbol{g}_{12} + \boldsymbol{g}_{22}}{2} + \frac{\boldsymbol{g}_{32} + \boldsymbol{g}_{22}}{2} + \frac{\boldsymbol{g}_{23} + \boldsymbol{g}_{22}}{2}}{4} + \frac{\boldsymbol{g}_{22}}{4}$$

$$\boldsymbol{g}'_{23} = \mathbf{E}_{23}$$

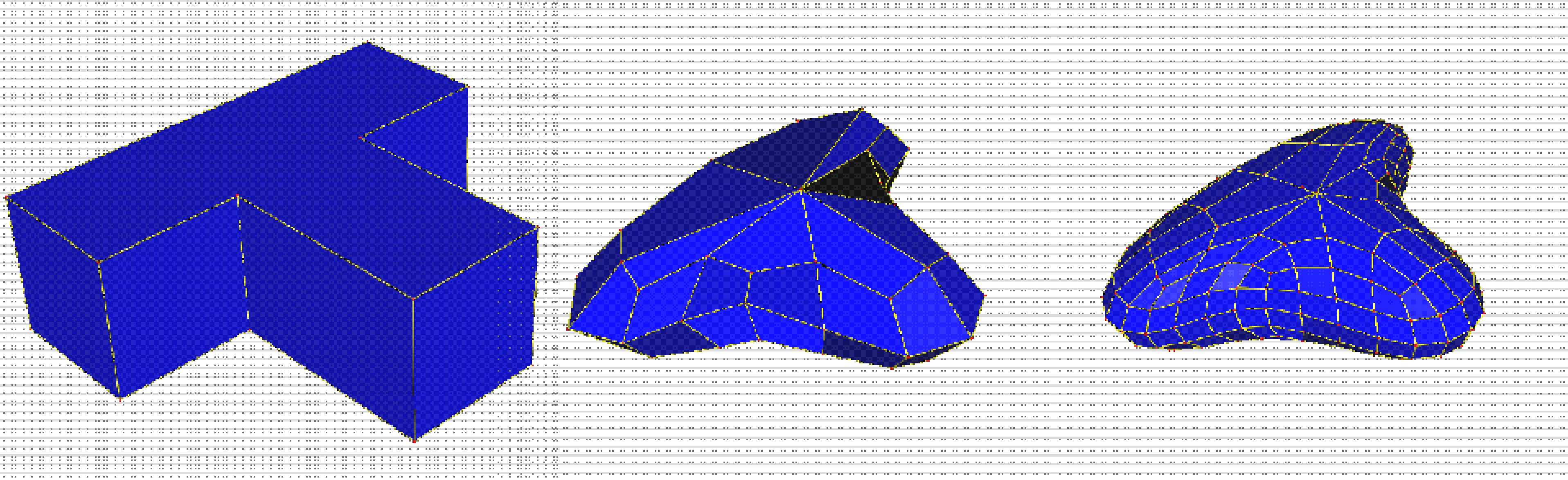
$$\boldsymbol{g}'_{24} = \frac{\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{22} + \mathbf{F}_{23}}{4} + \frac{\frac{\boldsymbol{g}_{22} + \boldsymbol{g}_{23}}{2} + \frac{\boldsymbol{g}_{13} + \boldsymbol{g}_{23}}{2} + \frac{\boldsymbol{g}_{33} + \boldsymbol{g}_{23}}{2} + \frac{\boldsymbol{g}_{24} + \boldsymbol{g}_{23}}{2}}{4} + \frac{\boldsymbol{g}_{23}}{4}$$

Catmull-Clack Surfaces

1. Construct the face points.
2. Construct the edge points.
3. Construct the single vertex point.
4. Connect the edges to the points.
    1. Connecting the face points to the edge points that correspond to edges on the face.
    2. Connecting the vertex point to the edge points.

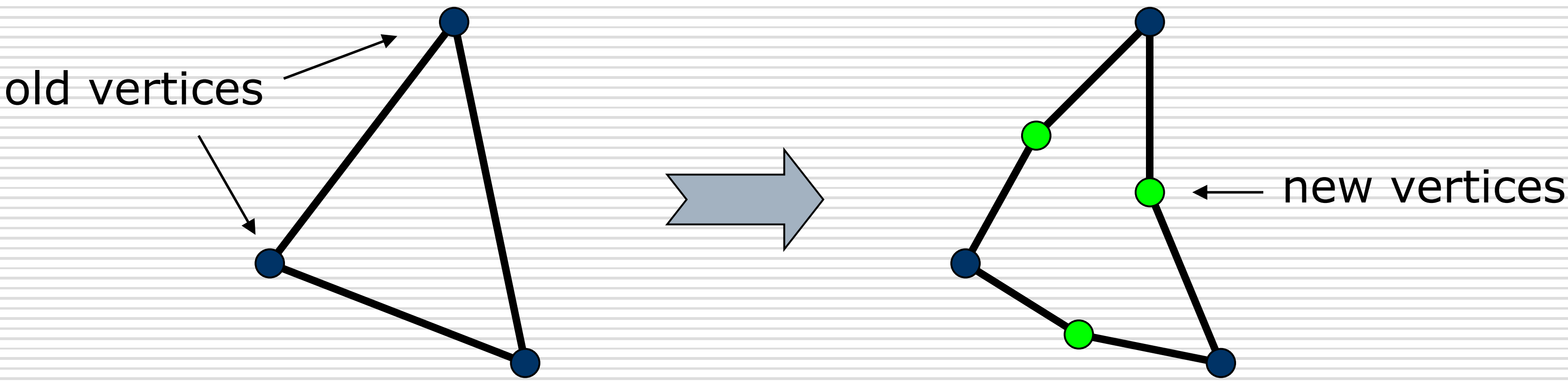


Catmull-Clack Surfaces



Triangular Subdivision

- Defined for triangular meshes (control nets)

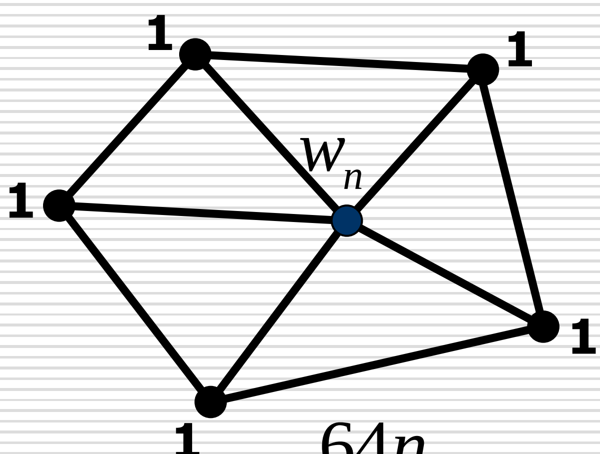


- Every face replaced by 4 new triangular faces
- Two kinds of new vertices:
  - Green vertices are associated with old edges
  - Blue vertices are associated with old vertices

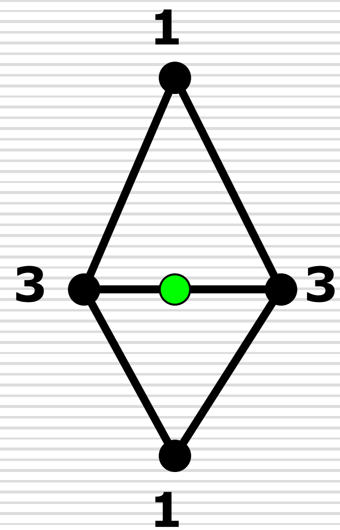
Loop’s Scheme

- New vertex is weighted average of old vertices.
- List of weights called subdivision *mask* or *stencil*.

rule for new blue vertices (n = vertex valence)

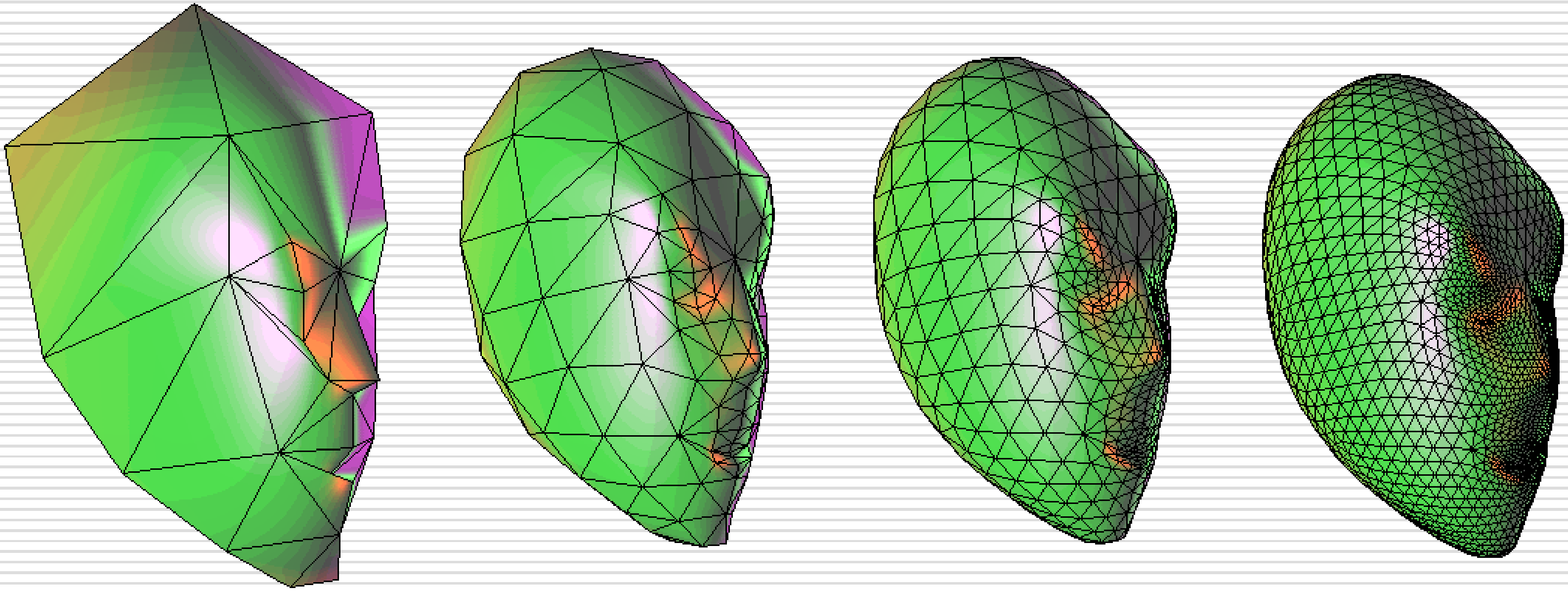


rule for new green vertices



$$w_n = \frac{64n}{40 - (3 + 2\cos(2\pi / n))^2} - n$$

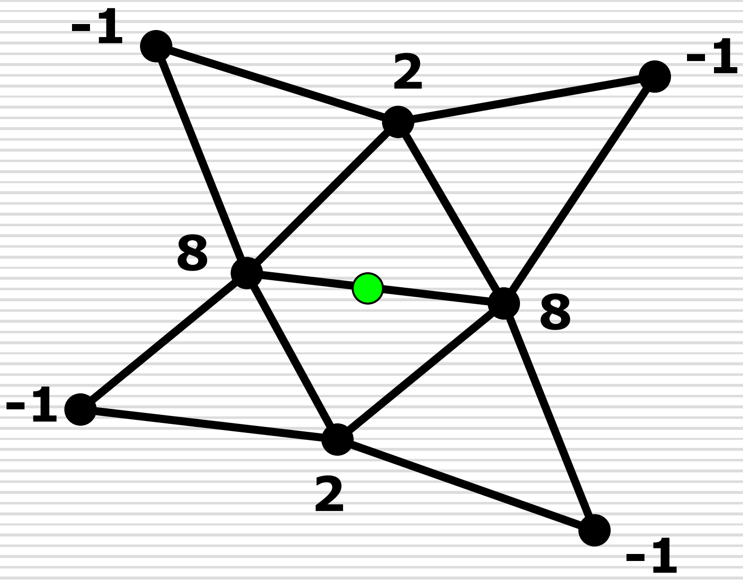
Example



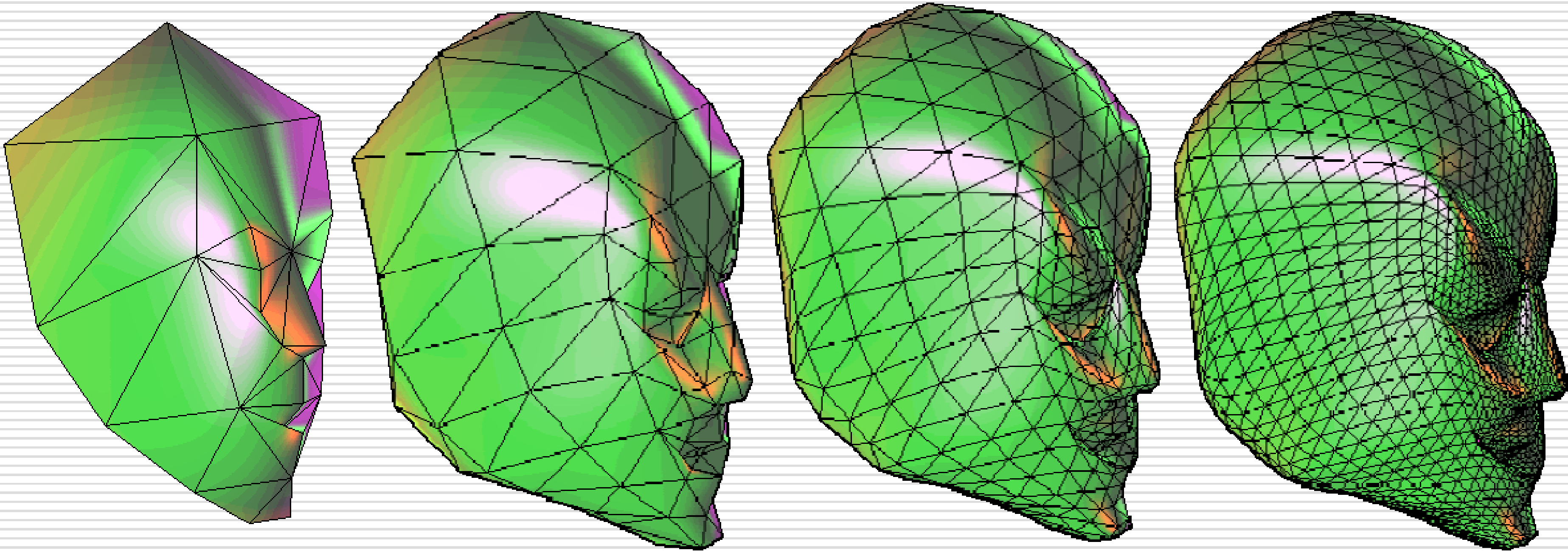
- Limit surface of Loop's subdivision is C^2 almost everywhere

Butterfly Scheme

- Interpolatory scheme
- New blue vertices inherit location of old vertices.
- New green vertices calculated by following *stencil*:

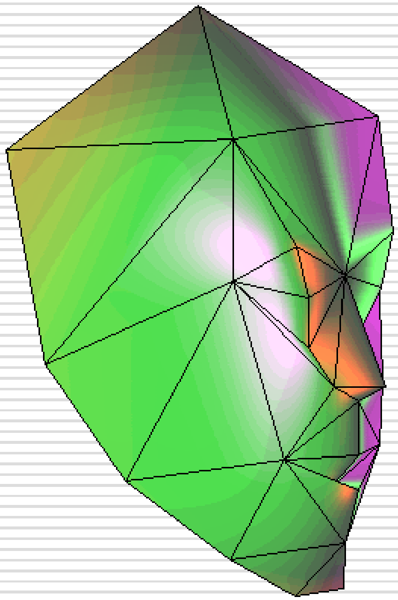


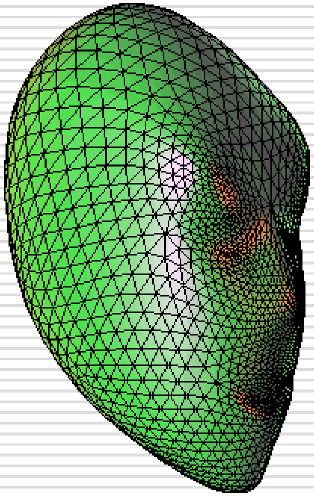
Example

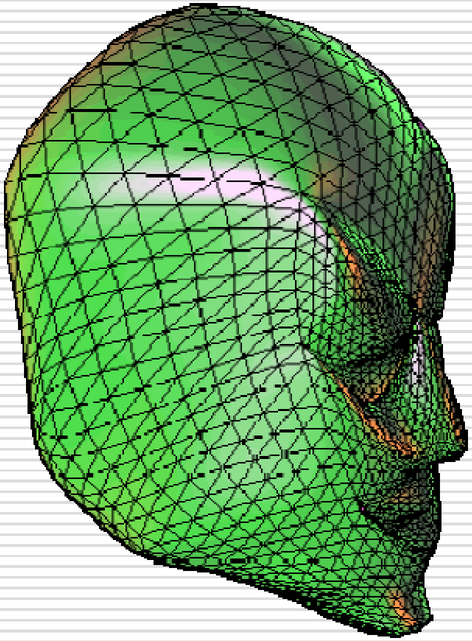


- Limit surfaces of Butterfly subdivision are C^1, but do not have second derivative
  - only at the edge whose end points are of degree 6

Comparison

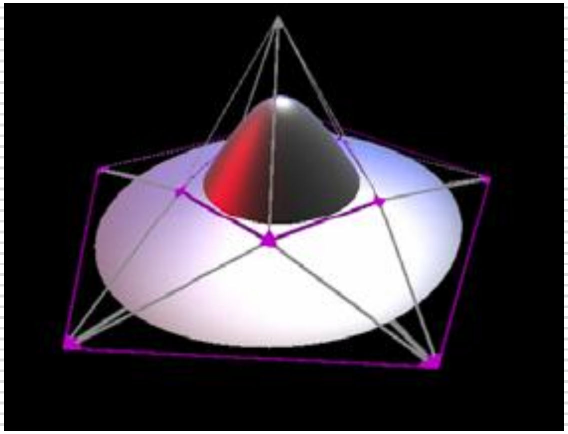


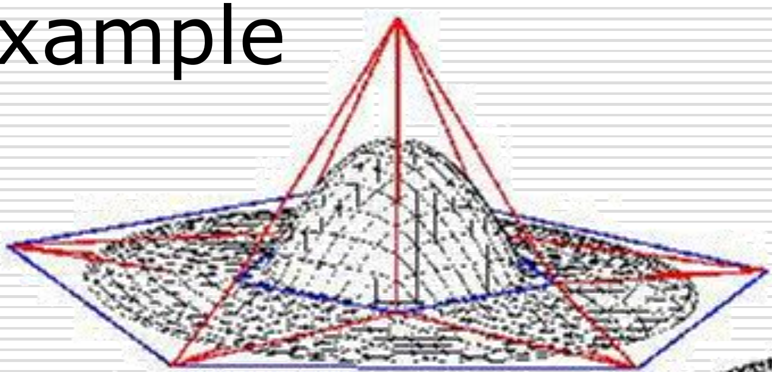


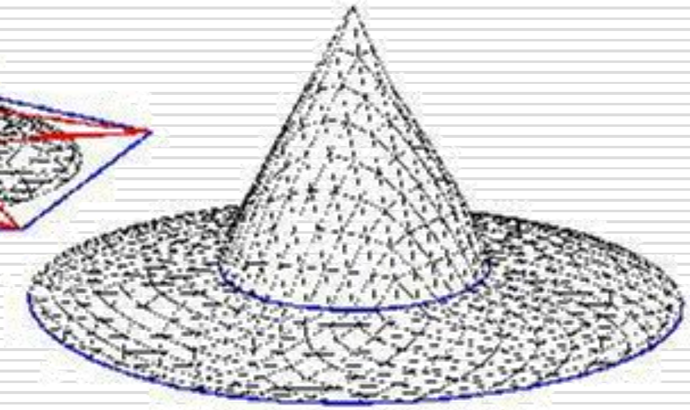


Boundaries & Creases

- special rules on and near the boundary
- boundary independent of the interior
- crease example







Adaptive Subdivision

- refine only if a criterion is satisfied
- advantages
  - potential savings in memory / time
- disadvantages
  - overhead
  - losses for “almost uniform” subdivision

Subdivision Schemes

face split

	Triangular meshes	Quad. meshes
approximating	Loop(C^2)	Catmull-Clark(C^2)
interpolating	Mod. Butterfly(C^1)	Kobbelt(C^1)

vertex split

Doo-Sabin, Midedge(C^1) Biquartic(C^2)